

A construction of the Glashow-Weinberg-Salam model on the lattice with exact gauge invariance

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ABSTRACT: We present a gauge-invariant and non-perturbative construction of the Glashow-Weinberg-Salam model on the lattice, based on the lattice Dirac operator satisfying the Ginsparg-Wilson relation. Our construction covers all $SU(2)$ topological sectors with vanishing $U(1)$ magnetic flux and would be usable for a description of the baryon number non-conservation. In infinite volume, it provides a gauge-invariant regularization of the electroweak theory to all orders of perturbation theory. First we formulate the reconstruction theorem which asserts that if there exists a set of local currents satisfying certain properties, it is possible to reconstruct the fermion measure which depends smoothly on the gauge fields and fulfills the fundamental requirements such as locality, gauge-invariance and lattice symmetries. Then we give a closed formula of the local currents required for the reconstruction theorem.

KEYWORDS: Lattice Gauge Field Theories, Anomalies in Field and String Theories, Gauge Symmetry, Standard Model.

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1. Introduction

There are several interesting possibilities in the dynamics of chiral gauge theories: fermion number non-conservation due to chiral anomaly [1, 2], various realizations of the gauge symmetry and global flavor symmetry [3, 4], the existence of massless composite fermions

suggested by 't Hooft's anomaly matching condition [5] and so on. Unfortunately, very little is known so far about the actual behavior of chiral gauge theories beyond perturbation theory. It is desirable to develop a formulation to study the non-perturbative aspect of chiral gauge theories.

Despite the well-known problem of the species doubling [6–9], lattice gauge theory can now provide a framework for non-perturbative formulation of chiral gauge theories. The clue to this development is the construction of local and gauge-covariant lattice Dirac operators satisfying the Ginsparg-Wilson relation [10–15]. By this relation, it is possible to realize an exact chiral symmetry on the lattice [16], without the species doubling problem. It is also possible to introduce Weyl fermions on the lattice and this opens the possibility to formulate anomaly-free chiral lattice gauge theories [17–29]. In the case of $U(1)$ chiral gauge theories, Lüscher [18] proved rigorously that it is possible to construct the fermion path-integral measure which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality, gauge-invariance and lattice symmetries. Although it is believed that a chiral gauge theory is a difficult case for numerical simulations because the effective action induced by Weyl fermions has a non-zero imaginary part, it would be still interesting and even useful to develop a formulation of chiral lattice gauge theories by which one can work out fermionic observables numerically as the functions of link field with exact gauge invariance.¹

In this article, we construct the $SU(2) \times U(1)$ chiral gauge theory of the Glashow-Weinberg-Salam model on the lattice, keeping the exact gauge invariance. As in the case of $U(1)$ theories, we first formulate the reconstruction theorem which asserts that if there exists a set of local currents satisfying certain properties, it is possible to reconstruct the chiral fermion measure which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality,² gauge-invariance and lattice symmetries.³ We then give a closed expression of the local currents (the fermion measure term) for the $SU(2) \times U(1)$ chiral lattice gauge theory defined on the finite-volume lattice. Our construction covers all $SU(2)$ topological sectors with vanishing $U(1)$ magnetic fluxes. This formulation provides the first gauge-invariant and non-perturbative regularization of the electroweak theory, which would be usable in both perturbative and non-perturbative analyses. In particular, it would be usable for a description of the baryon number non-conservation.⁴

This article is organized as follows. In section 2, we introduce our lattice formulation of the Glashow-Weinberg-Salam model at the classical level. In section 3, we define the

¹In the above formulation of $U(1)$ chiral lattice gauge theories [18], although the proof of the existence of the fermion measure is constructive, the resulted formula of the fermion measure turns out to be rather complicated for the case of the finite-volume lattice. It also relies on the results obtained in the infinite lattice. Therefore it does not provide a formulation which is immediately usable for numerical applications. See [33–35] for a simplified formulation toward a practical implementation.

²We adopt the generalized notion of locality on the lattice given in [15, 17, 18] for Dirac operators and composite fields. See also [33] for the case of the finite volume lattice.

³The lattice symmetries mean translations, rotations, reflections and charge conjugation.

⁴In the continuum theory, there is no winding number associated with the abelian gauge fields in four dimensions. Therefore, we believe that the construction in the $SU(2)$ topological sectors with vanishing $U(1)$ magnetic fluxes would be sufficient for a description of the baryon number non-conservation.

path-integral measure of chiral fermion fields and formulate the reconstruction theorem. In section 4, we give an explicit formula of the local currents (the measure term) which fulfills all the required properties for the reconstruction theorem. In section 5, we discuss the measure term in the infinite volume limit. Section 6 is devoted to discussions.

2. The Glashow-Weinberg-Salam model on the lattice

In this section, we describe a construction of the Glashow-Weinberg-Salam model on the lattice within the framework of chiral lattice gauge theories based on the lattice Dirac operator satisfying the Ginsparg-Wilson relation [18, 19]. We assume a local, gauge-covariant lattice Dirac operator D which satisfies the Ginsparg-Wilson relation. An explicit example of such lattice Dirac operator is given by the overlap Dirac operator [11, 13], which was derived from the overlap formalism [36–46].⁵ In this case, our formulation is equivalent to the overlap formalism for chiral lattice gauge theories⁶ or the domain wall fermion approach [58, 59]. See [60] for the attempt to construct the standard model in the domain wall fermion approach combined with the construction by Eichten and Preskill [61].⁷

2.1 $SU(2) \times U(1)$ Gauge fields

We consider the four-dimensional lattice of the finite size L and choose lattice units,

$$\Gamma = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid 0 \leq x_\mu < L (\mu = 1, 2, 3, 4)\}. \quad (2.1)$$

Adopting the compact formulation for $U(1)$ lattice gauge theory, the $SU(2)$ and $U(1)$ gauge fields on Γ may be represented through periodic link fields on the infinite lattice:

$$U^{(1)}(x, \mu) \in U(1), \quad x \in \mathbb{Z}^4, \quad (2.2)$$

$$U^{(1)}(x + L\hat{\nu}, \mu) = U^{(1)}(x, \mu) \quad \text{for all } \mu, \nu, \quad (2.3)$$

and

$$U^{(2)}(x, \mu) \in SU(2), \quad x \in \mathbb{Z}^4, \quad (2.4)$$

$$U^{(2)}(x + L\hat{\nu}, \mu) = U^{(2)}(x, \mu) \quad \text{for all } \mu, \nu. \quad (2.5)$$

We require the so-called admissibility condition on the gauge fields,

$$|F_{\mu\nu}(x)| < \epsilon_1, \quad F_{\mu\nu}(x) \equiv \frac{1}{i} \ln P^{(1)}(x, \mu, \nu) \in (-\pi, \pi), \quad (2.6)$$

$$\|1 - P^{(2)}(x, \mu, \nu)\| < \epsilon_2, \quad (2.7)$$

⁵The overlap formula was derived from the five-dimensional approach of domain wall fermion proposed by Kaplan [47]. In the vector-like formalism of domain wall fermion [48–51], the local low energy effective action of the chiral mode precisely reproduces the overlap Dirac operator [52–54].

⁶The overlap formalism gives a well-defined partition function of Weyl fermions on the lattice, which nicely reproduces the fermion zero mode and the fermion-number violating observables ('t Hooft vertices) [55–57]. The gauge-invariant construction by Lüscher [18] provides a procedure to fix the ambiguity of the complex phase of the overlap formula in a gauge-invariant manner for anomaly-free $U(1)$ chiral gauge theories.

⁷See also [62–67] for the recent attempt to construct chiral gauge theories using mirror Ginsparg-Wilson fermions with gauge- and chiral-invariant Yukawa couplings to the extra bosonic degrees of freedom, which may be identified with the Higgs field or Wess-Zumino scalar field, and for related works.

for all x, μ, ν , where the plaquette variables are defined by

$$P^{(i)}(x, \mu, \nu) = U^{(i)}(x, \mu)U^{(i)}(x + \hat{\mu}, \nu)U^{(i)}(x + \hat{\nu}, \mu)^{-1}U^{(i)}(x, \nu)^{-1} \quad (i = 1, 2). \quad (2.8)$$

This condition ensures that the overlap Dirac operator [11, 13] is a smooth and local function of the gauge field if $(Y\epsilon_1) < 1/30$ and $\epsilon_2 < 1/30$, where Y is the hyper-charge of the fermion on which the overlap Dirac operator acts [15].

To impose the admissibility condition dynamically, we adopt the following action for the gauge fields:

$$S_G = \frac{1}{g_2^2} \sum_{x \in \Gamma} \sum_{\mu, \nu} \text{tr}\{1 - P^{(2)}(x, \mu, \nu)\} \left[1 - \text{tr}\{1 - P^{(2)}(x, \mu, \nu)\} / \epsilon_2^2\right]^{-1} + \frac{1}{4g_1^2} \sum_{x \in \Gamma} \sum_{\mu, \nu} [F_{\mu\nu}(x)]^2 \left\{1 - [F_{\mu\nu}(x)]^2 / \epsilon_1^2\right\}^{-1}. \quad (2.9)$$

2.2 Quarks and leptons

Right- and left- handed Weyl fermions are introduced on the lattice based on the Ginsparg-Wilson relation. Let us first consider a generic gauge group G and a Dirac field $\psi(x)$ coupled to the gauge field $U(x, \mu)$ in a certain representation R of G . Then we assume a local, gauge-covariant lattice Dirac operator D_L which acts on $\psi(x)$ and satisfies the Ginsparg-Wilson relation,

$$\gamma_5 D_L + D_L \gamma_5 = 2D_L \gamma_5 D_L. \quad (2.10)$$

The kernel of the lattice Dirac operator in finite volume, D_L , may be represented through the kernel of the lattice Dirac operator in infinite volume, D , as follows:

$$D_L(x, y) = D(x, y) + \sum_{n \in \mathbb{Z}^4, n \neq 0} D(x, y + nL), \quad (2.11)$$

where $D(x, y)$ is defined with a periodic link field in infinite volume. We assume that $D(x, y)$ possesses the locality property given by

$$\|D(x, y)\| \leq C(1 + \|x - y\|^p) e^{-\|x - y\|/\varrho} \quad (2.12)$$

for some constants $\varrho > 0$, $C > 0$, $p \geq 0$, where ϱ is the localization range of the lattice Dirac operator.

Given such a lattice Dirac operator D_L , one can introduce a chiral operator as

$$\hat{\gamma}_5 \equiv \gamma_5(1 - 2D_L), \quad (\hat{\gamma}_5)^2 = \mathbb{I}. \quad (2.13)$$

Then, the right- and left-handed Weyl fermions in the representation R of G can be defined by the eigenstates of the chiral operator $\hat{\gamma}_5$ (and γ_5 for the anti-fields). Namely,

$$\psi_{\pm}(x) = \hat{P}_{\pm} \psi(x), \quad \bar{\psi}_{\pm}(x) = \bar{\psi}(x) P_{\mp}, \quad (2.14)$$

where \hat{P}_{\pm} and P_{\pm} are the chiral projection operators given by

$$\hat{P}_{\pm} = \left(\frac{1 \pm \hat{\gamma}_5}{2}\right), \quad P_{\pm} = \left(\frac{1 \pm \gamma_5}{2}\right). \quad (2.15)$$

Now we consider quarks and leptons in the Glashow-Weinberg-Salam model. For simplicity, we consider the first family. We adopt the convention for the normalization of the hyper-charges such that the Nishijima-Gell-Mann relation reads $Q = T_3 + \frac{1}{6}Y$. To describe the left-handed quarks and leptons, which are SU(2) doublets, we introduce a left-handed fermion $\psi_-(x)$ with the index $\alpha(= 1, \dots, 4)$, each component of which couples to the SU(2) \times U(1) gauge field, $U^{(2)}(x, \mu) \otimes \{U^{(1)}(x, \mu)\}^{Y_\alpha}$, with the hyper-charge Y_α ($Y_{1,2,3} = 1$ and $Y_4 = -3$). Namely,

$$\psi_-(x) = {}^t (q_-^1(x), q_-^2(x), q_-^3(x), l_-(x)). \quad (2.16)$$

Similarly, to describe the right-handed quarks and leptons, which are SU(2) singlets, we introduce a right-handed fermion $\psi_+(x)$ with the index $\beta(= 1, \dots, 8)$, each component of which couples to the U(1) gauge field, $\{U^{(1)}(x, \mu)\}^{Y_\beta}$, with the hyper-charge Y_β ($Y_{1,3,5} = 4$, $Y_{2,4,6} = -2$, $Y_7 = 0$ and $Y_8 = -6$). Namely,

$$\psi_+(x) = {}^t (u_+^1(x), d_+^1(x), u_+^2(x), d_+^2(x), u_+^3(x), d_+^3(x), \nu_+(x), e_+(x)). \quad (2.17)$$

Then the action of quarks and leptons is given by

$$S_F = \sum_{x \in \Gamma} \bar{\psi}_-(x) D_L \psi_-(x) + \sum_{x \in \Gamma} \bar{\psi}_+(x) D_L \psi_+(x). \quad (2.18)$$

2.3 Higgs field and its Yukawa-couplings to quarks and leptons

Higgs field is a SU(2) doublet with the hyper-charge $Y_h = +6$. The action of the Higgs field may be given by

$$S_H = \sum_x \left[\sum_\nu (\nabla_\nu \phi(x))^\dagger \nabla_\nu \phi(x) + \frac{\lambda}{2} (\phi(x)^\dagger \phi(x) - v^2)^2 \right], \quad (2.19)$$

where $\phi(x)$ couples to the gauge field $U^{(2)}(x, \mu) \otimes \{U^{(1)}(x, \mu)\}^{Y_h}$ and ∇_ν is the SU(2) \times U(1) gauge-covariant difference operator. Yukawa couplings of the Higgs field to the quarks and leptons may also be introduced as follows:⁸

$$S_Y = \sum_x \left[y_u \bar{q}_-^i(x) \tilde{\phi}(x) u_+^i(x) + y_u^* \bar{u}_+^i(x) \tilde{\phi}(x)^\dagger q_-^i(x) \right. \\ \left. + y_d \bar{q}_-^i(x) \phi(x) d_+^i(x) + y_d^* \bar{d}_+^i(x) \phi(x)^\dagger q_-^i(x) \right. \\ \left. + y_l \bar{l}_-(x) \phi(x) e_+(x) + y_l^* \bar{e}_+(x) \phi(x)^\dagger l_-(x) \right], \quad (2.20)$$

where $\tilde{\phi}(x)$ is the SU(2) conjugate of $\phi(x)$.

Thus the total lattice action,

$$S = S_G + S_F + S_H + S_Y, \quad (2.21)$$

defines a classical theory of the Glashow-Weinberg-Salam model [30–32] on the lattice with the first-family quarks and leptons. In this action, locality, gauge-invariance and lattice

⁸One may add the Dirac-type mass term for the neutrino, $\sum_x \{y_\nu \bar{l}_-(x) \tilde{\phi}(x) \nu_+(x) + y_\nu^* \bar{\nu}_+(x) \tilde{\phi}(x)^\dagger l_-(x)\}$.

symmetries such as translations and rotations are manifest. CP symmetry, however, is not manifest even with the real Yukawa couplings. But it is possible to show that at the quantum level both the partition function and the on-shell amplitudes respect the CP symmetry [70–72, 68]. With the three families, then, the breaking of CP symmetry comes from the Kobayashi-Maskawa phase [69] as in the continuum theory.

2.4 Topology of the $SU(2) \times U(1)$ gauge fields

The admissibility condition ensures that the overlap Dirac operator [11, 13] is a smooth and local function of the gauge field [15]. Then, through the lattice Dirac operator D_L , it is possible to define a topological charge of the gauge fields [37, 38, 40, 12, 16]: for the admissible $SU(2)$ and $U(1)$ gauge fields, one has

$$Q^{(i)} = \text{Tr} \gamma_5 (1 - D_L)|_{U=U^{(i)}} = \sum_{x \in \Gamma} \text{tr} \{ \gamma_5 (1 - D_L) \} (x, x)|_{U=U^{(i)}} \quad (i = 1, 2), \quad (2.22)$$

where $D_L(x, y)$ is the kernel of the lattice Dirac operator D_L . For $0 < \epsilon_1 < \pi/3$, the admissible $U(1)$ gauge fields can also be classified by the magnetic fluxes,

$$m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t=0}^{L-1} F_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu}), \quad (2.23)$$

which are integers independent of x . $m_{\mu\nu}$ is related to $Q^{(1)}$ by $Q^{(1)} = (1/2) \sum_{\mu\nu} m_{\mu\nu}^2$ [73]. Then the admissible $SU(2)$ and $U(1)$ gauge fields can be classified by the topological numbers Q_2 and $m_{\mu\nu}$, respectively.⁹ We denote the space of the admissible $SU(2)$ gauge fields with a given topological charge $Q^{(2)}$ by $\mathfrak{U}^{(2)}[Q]$ and the space of the admissible $U(1)$ gauge fields with a given magnetic fluxes $m_{\mu\nu}$ by $\mathfrak{U}^{(1)}[m]$.

3. Path-integral measure of the lattice Glashow-Weinberg-Salam model

In this section, we consider a construction of the path-integral measure of the quarks and leptons in the lattice Glashow-Weinberg-Salam model.¹⁰ We will show that, as in the case of the $U(1)$ chiral gauge theories [18], it is possible to formulate a reconstruction theorem of the fermion measure for the topological sectors of the admissible $SU(2) \times U(1)$ gauge fields with vanishing $U(1)$ magnetic fluxes. This reconstruction theorem asserts that if there exist local currents which satisfy certain properties, it is possible to reconstruct the fermion measure which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality, gauge-invariance and lattice symmetries.

⁹Strictly speaking, the complete topological classification of the space of admissible $SU(2)$ gauge fields is not known yet. However, as we will see, our construction is valid for any $SU(2)$ topological sectors, as long as the $U(1)$ magnetic flux vanishes identically.

¹⁰The path-integral measure of the $SU(2) \times U(1)$ gauge fields and Higgs field may be defined as usual.

3.1 Path-integral measure of Quarks and Leptons

The path-integral measure of quark fields and lepton fields may be defined by the Grassmann integrations,

$$\mathcal{D}[\psi_+] \mathcal{D}[\bar{\psi}_+] \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] = \prod_j db_j \prod_k d\bar{b}_k \prod_j dc_j \prod_k d\bar{c}_k, \quad (3.1)$$

where $\{b_j, \bar{b}_k\}$ and $\{c_j, \bar{c}_k\}$ are the Grassman coefficients in the expansion of the chiral fields,

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar{\psi}_+(x) = \sum_k \bar{b}_k \bar{u}_k(x), \quad (3.2)$$

$$\psi_-(x) = \sum_j v_j(x) c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x), \quad (3.3)$$

in terms of the chiral (orthonormal) bases defined by

$$\hat{P}_+ u_j(x) = u_j(x), \quad \bar{u}_k(x) P_- = \bar{u}_k(x). \quad (3.4)$$

$$\hat{P}_- v_j(x) = v_j(x), \quad \bar{v}_k(x) P_+ = \bar{v}_k(x). \quad (3.5)$$

Since the projection operators \hat{P}_\pm depend on the gauge fields through D , the fermion measure also depends on the gauge fields.

This gauge field dependence can be examined explicitly by considering the effective action induced by the quarks and leptons,

$$\Gamma_{\text{eff}} = \ln [\det(\bar{v}_k D_L v_j) \det(\bar{u}_k D_L u_j)]. \quad (3.6)$$

With respect to the variation of the gauge fields,

$$\delta_\eta U^{(1)}(x, \mu) = i\eta_\mu^{(1)}(x) U^{(1)}(x, \mu), \quad (3.7)$$

$$\delta_\eta U^{(2)}(x, \mu) = i\eta_\mu^{(2)}(x) U^{(2)}(x, \mu), \quad (\eta_\mu^{(2)}(x) \equiv \eta_\mu^a(x) T^a), \quad (3.8)$$

the variation of the effective action Γ_{eff} is evaluated as

$$\delta_\eta \Gamma_{\text{eff}} = \text{Tr} \left\{ \delta_\eta D_L \hat{P}_- D_L^{-1} P_+ \right\} + \text{Tr} \left\{ \delta_\eta D_L \hat{P}_+ D_L^{-1} P_- \right\} + \sum_j (v_j, \delta_\eta v_j) + \sum_j (u_j, \delta_\eta u_j). \quad (3.9)$$

In particular, for the gauge transformations

$$\eta_\mu^{(1)}(x) = -\partial_\mu \omega(x), \quad (3.10)$$

$$\eta_\mu^{(2)}(x) = -[\nabla_\mu \omega]^a(x) T^a, \quad (3.11)$$

it is given as

$$\begin{aligned} \delta_\omega \Gamma_{\text{eff}} &= i \sum_{x \in \Gamma} \omega(x) [\text{tr} \{ Y_- \gamma_5 (1 - D_L)(x, x) \} - \text{tr} \{ Y_+ \gamma_5 (1 - D_L)(x, x) \}] \\ &+ i \sum_{x \in \Gamma} \omega^a(x) \text{tr} \{ T^a \gamma_5 (1 - D_L)(x, x) \} + \sum_j (v_j, \delta_\omega v_j) + \sum_j (u_j, \delta_\omega u_j), \end{aligned} \quad (3.12)$$

where $Y_- = \text{diag}(1, 1, 1, -3)$ and $Y_+ = \text{diag}(4, -2, \dots, 0, -6)$.¹¹

In this gauge-field dependence of the fermion measure, there is an ambiguity by a pure phase factor, because any unitary transformations of the bases,

$$\tilde{u}_j(x) = \sum_l u_l(x) (\mathcal{Q}_+^{-1})_{lj}, \quad \tilde{b}_j = \sum_l (\mathcal{Q}_+)_{jl} b_l, \quad (3.13)$$

$$\tilde{v}_j(x) = \sum_l v_l(x) (\mathcal{Q}_-^{-1})_{lj}, \quad \tilde{c}_j = \sum_l (\mathcal{Q}_-)_{jl} c_l, \quad (3.14)$$

induces a change of the measure by the pure phase factor $\det \mathcal{Q}_+ \cdot \det \mathcal{Q}_-$. This ambiguity should be fixed so that the measure fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries.

3.2 Gauge anomaly cancellations in the lattice Glashow-Weinberg-Salam model

We next examine the gauge anomaly cancellations in the lattice Glashow-Weinberg-Salam model.

3.2.1 Pseudo reality of SU(2) and the absence of SU(2)³ gauge anomaly

We first consider the case where the U(1) link field is trivial. In the topological sectors with vanishing U(1) magnetic flux, $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$, any admissible U(1) link field can be continuously deformed to the trivial configuration, $U^{(1)}(x, \mu) = 1$. In this limit, only the SU(2) gauge field couples to the left-handed fermion $\psi_-(x)$, which now consists of four degenerate SU(2) doublets. By noting the pseudo reality of SU(2),

$$U^{(2)}(x, \mu)^* = (i\sigma_2) U^{(2)}(x, \mu) (i\sigma_2)^{-1}, \quad (3.15)$$

and the charge- and γ_5 -conjugation properties of the lattice Dirac operator,

$$D_L[U^{(2)*}] = C^{-1} \{D_L[U^{(2)}]\}^T C, \quad D_L[U^{(2)}]^\dagger = \gamma_5 D_L[U^{(2)}] \gamma_5, \quad (3.16)$$

where C is the charge conjugation matrix satisfying $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, one can infer that

$$D_L[U^{(2)}] = (\gamma_5 C^{-1} \otimes i\sigma_2) \{D_L[U^{(2)}]\}^* (C\gamma_5 \otimes (i\sigma_2)^{-1}). \quad (3.17)$$

Then one may choose the basis vectors of the left-handed fermion $\psi_-(x) = {}^t(q_-^1(x), q_-^2(x), q_-^3(x), l_-(x))$ for any given SU(2) gauge field $U^{(2)}(x, \mu) \in \mathfrak{U}^{(2)}[Q]$ as follows:

$$q_-^1(x) = \sum_j w_j(x) c_j^1, \quad (3.18)$$

$$q_-^2(x) = \sum_j (\gamma_5 C^{-1} \otimes i\sigma_2) [w_j(x)]^* c_j^2, \quad (3.19)$$

$$q_-^3(x) = \sum_j w_j(x) c_j^3, \quad (3.20)$$

$$l_-(x) = \sum_j (\gamma_5 C^{-1} \otimes i\sigma_2) [w_j(x)]^* c_j^4, \quad (3.21)$$

¹¹Throughout this paper, Tr stands for the trace over the lattice index $x (\in \Gamma)$, the flavor indices α, β and the spinor index. tr stands for the trace over the flavor and/or spinor indices only.

where $\{w_j(x)\}$ is an arbitrarily chosen basis for a single left-handed SU(2) doublet. With this choice of the basis, one can infer that the measure term vanishes identically and therefore the fermion measure is manifestly invariant under the SU(2) gauge transformation, eqs. (3.8) and (3.11).

3.2.2 Cancellations of SU(2)² × U(1) and U(1)³ gauge anomalies

When the U(1) link field is non-trivial in generic topological sectors, $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[m]$, the U(1) part of the gauge anomaly is given by

$$q_L^{(1)}(x) = \text{tr}\{Y_- \gamma_5 (1 - D_L)(x, x)\} - \text{tr}\{Y_+ \gamma_5 (1 - D_L)(x, x)\}, \quad (3.22)$$

where $D_L(x, y)$ is the finite-volume kernel of the lattice Dirac operator. It is topological in the sense that

$$\sum_{x \in \Gamma} q_L^{(1)}(x) = \text{integer}, \quad \sum_{x \in \Gamma} \delta_\eta q_L^{(1)}(x) = 0. \quad (3.23)$$

Then the following lemma holds true concerning the cancellations of SU(2)² × U(1) and U(1)³ gauge anomalies:

Lemma 1. *In the lattice Glashow-Weinberg-Salam model, the U(1) gauge anomaly has the following form in sufficiently large volume L⁴:*

$$\begin{aligned} q_L^{(1)}(x) = & \text{tr}\{Y_- \gamma_5 (1 - D_L)(x, x)\}|_{U=U^{(2)}} \\ & + (\text{tr}\{Y_-^3\} - \text{tr}\{Y_+^3\}) \gamma \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x + \hat{\mu} + \hat{\nu}) \\ & + \partial_\mu^* k_\mu(x), \end{aligned} \quad (3.24)$$

where γ is a constant independent of the gauge fields and $k_\mu(x)$ is a local, gauge-invariant current which can be constructed so that it transforms as an axial vector current under the lattice symmetries. Moreover, since the hyper-charges of a single family of quarks and leptons satisfy the anomaly cancellation conditions,

$$\text{tr}\{Y_-^3\} = 0, \quad (3.25)$$

$$\text{tr}\{Y_-^3\} - \text{tr}\{Y_+^3\} = 0, \quad (3.26)$$

the cohomologically non-trivial part of the gauge anomaly cancels exactly at a finite lattice spacing and the total U(1) gauge anomaly is cohomologically trivial:

$$q_L^{(1)}(x) = \partial_\mu^* k_\mu(x). \quad (3.27)$$

Proof. Given the topological property of the U(1) gauge anomaly, it is possible to apply the cohomology-analysis developed for the U(1) case [17, 83, 73, 84] to the SU(2) × U(1) case by regarding the SU(2) gauge field $U^{(2)}(x, \mu)$ as a background.¹² The result is given

¹²This trick was first used to show the gauge anomaly cancellation in the lattice Glashow-Weinberg-Salam model [28] in the 4+2 dimensional approach to the cohomological analysis of non-abelian gauge anomalies [19, 24, 85].

by the following expression:

$$\begin{aligned}
 q_L^{(1)}(x) &= \text{tr}\{Y_- \gamma_5 (1 - D_L)(x, x)\}|_{U=U(2)} \\
 &\quad + \beta_{\mu\nu}(x) F_{\mu\nu}(x) \\
 &\quad + (\text{tr}\{Y_-^3\} - \text{tr}\{Y_+^3\}) \gamma \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x + \hat{\mu} + \hat{\nu}) \\
 &\quad + \partial_\mu^* k_\mu(x),
 \end{aligned} \tag{3.28}$$

where γ is a constant independent of the gauge fields, which takes the value $\gamma = \frac{1}{32\pi^2}$ for the overlap Dirac operator [78], $\beta_{\mu\nu}(x)$ is a tensor field satisfying $\partial_\mu^* \beta_{\mu\nu}(x) = 0$ which depends only on the SU(2) gauge field and $k_\mu(x)$ is a local, gauge-invariant current which can be constructed so that it transforms as the axial vector current under the lattice symmetries. Moreover, taking into account the pseudo-scalar nature of $q_L^{(1)}(x)$ under the charge conjugation and the pseudo reality of SU(2), one has

$$q_L^{(1)}(x) \Big|_{U=U(2), U(1)^*} = q_L^{(1)}(x) \Big|_{U=U(2), U(1)}, \tag{3.29}$$

which immediately implies that the second term in the r.h.s. of eq. (3.28) can be included into the total-divergence term as

$$2 \beta_{\mu\nu}(x) F_{\mu\nu}(x) = \partial_\mu^* \left[k_\mu(x) \Big|_{U=U(2), U(1)^*} - k_\mu(x) \Big|_{U=U(2), U(1)} \right]. \tag{3.30}$$

We emphasize that this is the result of the U(1) gauge anomaly *in finite volume*, which is obtained by combining the result in the infinite lattice [17, 83, 73] with the use of the trick to regard the SU(2) gauge field $U^{(2)}(x, \mu)$ as a background [28], and the result of the analysis of the finite volume correction [84]. See also [33]. In fact, the local, gauge-invariant current $k_\mu(x)$ may be decomposed as

$$k_\mu(x) = \bar{k}_\mu(x) + \Delta k_\mu(x), \tag{3.31}$$

where $\bar{k}_\mu(x)$ and $\Delta k_\mu(x)$ satisfy the anomalous conservation laws,

$$\begin{aligned}
 \partial_\mu^* k_\mu(x) &= \text{tr}\{Y_- \gamma_5 (1 - D)(x, x)\} - \text{tr}\{Y_+ \gamma_5 (1 - D)(x, x)\} \\
 &\equiv q^{(1)}(x),
 \end{aligned} \tag{3.32}$$

$$\begin{aligned}
 \partial_\mu^* \Delta k_\mu(x) &= \sum_{n \in \mathbb{Z}^4, n \neq 0} [\text{tr}\{Y_- \gamma_5 (1 - D)(x, x + Ln)\} - \text{tr}\{Y_+ \gamma_5 (1 - D)(x, x + Ln)\}] \\
 &\equiv r(x),
 \end{aligned} \tag{3.33}$$

respectively. $\bar{k}_\mu(x)$ is obtained as the solution of the cohomology-analysis [17, 83, 73] applied to $q^{(1)}(x)$ in infinite volume, while $\Delta k_\mu(x)$ is the result of the analysis of the finite volume correction [84] applied to $r(x)$, both in the use of the trick to regard the SU(2) gauge field $U^{(2)}(x, \mu)$ as a background [28]. One can infer from eq. (2.12) that

$$|\Delta k_\mu(x)| \leq C_1 e^{-L/\rho} \tag{3.34}$$

for a constant $C_1 > 0$ [84]. This result should be compared with the result obtained from the 4+2 dimensional approach to the cohomological analysis of non-abelian gauge anomalies [28], where only the solution in the infinite volume limit has been obtained so far.

3.2.3 Issue related to SU(2) global anomaly

When the U(1) link field is trivial in the topological sectors with vanishing U(1) magnetic flux, $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$, one can construct the fermion measure which is invariant under the SU(2) gauge transformation, eqs. (3.8) and (3.11). However, there remains the issue related to SU(2) global anomaly [75–77]. In the following sections, we will establish rigorously that the lattice counterpart of the SU(2) global anomaly [75–77] is absent in the topological sectors with vanishing U(1) magnetic flux, $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$.

3.3 Reconstruction theorem of the fermion measure

We now formulate the reconstruction theorem of the fermion measure in the lattice Glashow-Weinberg-Salam model. The properties of the fermion measure can be characterized by the so-called measure term which is given in terms of the chiral basis and its variation with respect to the gauge fields as

$$\mathfrak{L}_\eta = i \sum_j (v_j, \delta_\eta v_j) + i \sum_j (u_j, \delta_\eta u_j). \quad (3.35)$$

Similar to the case of U(1) chiral lattice gauge theories [18], one can establish the following theorem.

Theorem. *In the topological sectors with vanishing U(1) magnetic flux, $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$, if there exist local currents $j_\mu^a(x)$ ($a = 1, 2, 3$), $j_\mu(x)$ which satisfy the following four properties, it is possible to reconstruct the fermion measure (the bases $\{u_j(x)\}$, $\{v_j(x)\}$) which depends smoothly on the gauge fields and fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries:*

1. $j_\mu^a(x)$, $j_\mu(x)$ are defined for all admissible SU(2) \times U(1) gauge fields in the given topological sectors and depends smoothly on the link variables.
2. $j_\mu^a(x)$ and $j_\mu(x)$ are gauge-covariant and -invariant, respectively and both transform as axial vector currents under the lattice symmetries.
3. The linear functional $\mathfrak{L}_\eta = \sum_{x \in \Gamma} \{\eta_\mu^a(x) j_\mu^a(x) + \eta_\mu(x) j_\mu(x)\}$ is a solution of the integrability condition

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + \mathfrak{L}_{[\eta, \zeta]} = i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + i \text{Tr} \left\{ \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} \quad (3.36)$$

for all periodic variations $\eta_\mu^a(x)$, $\eta_\mu(x)$ and $\zeta_\mu^a(x)$, $\zeta_\mu(x)$.

4. The anomalous conservation laws hold:

$$\{\nabla_\mu^* j_\mu\}^a(x) = \text{tr}\{T^a \gamma_5 (1 - D)(x, x)\}, \quad (3.37)$$

$$\partial_\mu^* j_\mu(x) = \text{tr}\{Y_- \gamma_5 (1 - D_L)(x, x)\} - \text{tr}\{Y_+ \gamma_5 (1 - D_L)(x, x)\}, \quad (3.38)$$

where $Y_- = \text{diag}(1, 1, 1, -3)$ and $Y_+ = \text{diag}(4, -2, \dots, 0, -6)$.

A comment is in order about the topological aspects of the reconstruction theorem. It is possible, as discussed in [18], to associate a U(1) bundle with the fermion measure. In this point of view, the measure term, \mathfrak{L}_η defined by eq. (3.35), can be regarded as the connection of the U(1) bundle, and the quantity which appears in the r.h.s. of the integrability condition eq. (3.36),

$$\mathfrak{C}_{\eta\zeta} \equiv i\text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + i\text{Tr} \left\{ \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} \quad (3.39)$$

is nothing but the curvature of the connection,

$$\mathfrak{C}_{\eta\zeta} = \delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + \mathfrak{L}_{[\eta,\zeta]}. \quad (3.40)$$

It is known that the integration of the curvature of a U(1) bundle over any two-dimensional closed surface in the base manifold takes value of the multiples of 2π . If one parametrizes a two-dimensional closed surface in the space of the admissible U(1) gauge fields by $s, t \in [0, 2\pi]$, then one has

$$\int_0^{2\pi} ds \int_0^{2\pi} dt \left[i\text{Tr} \left\{ \hat{P}_- [\partial_s \hat{P}_-, \partial_t \hat{P}_-] \right\} + i\text{Tr} \left\{ \hat{P}_+ [\partial_s \hat{P}_+, \partial_t \hat{P}_+] \right\} \right] = 2\pi \times \text{integer}. \quad (3.41)$$

If (and only if) the U(1) bundle is trivial, these integrals of the curvature vanish identically. The integrability condition eq. (3.36) asserts that it is indeed the case and the fermion measure is then smooth. The global integrability condition discussed in the next subsection, on the other hand, asserts that the holonomy of the U(1) bundle is reproduced by the "Wilson line" of the connection.

3.4 Proof of the reconstruction theorem

3.4.1 Global integrability condition

As a first step to prove the reconstruction theorem, we formulate the so-called global integrability condition [19].

Let us assume that currents $j_\mu^a(x)$ ($a = 1, 2, 3$) and $j_\mu(x)$ are local and satisfy all four properties required for the reconstruction theorem. We consider a definite topological sector $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ and choose an arbitrary reference field $U_0^{(2)} \otimes U_0^{(1)}$ in this sector. Any other field $U^{(2)} \otimes U^{(1)}$ in the same sector can then be reached through a smooth curve U_t such that $U_1 = U^{(2)} \otimes U^{(1)}$. Then the basis vectors of the fermion fields at the point $U^{(2)} \otimes U^{(1)}$ may be chosen as follows [19]:

$$v_j(x) = \begin{cases} Q_{1-v_1^0} W^{-1} & \text{if } j = 1, \\ Q_{1-v_j^0} & \text{otherwise,} \end{cases} \quad (3.42)$$

$$u_j(x) = Q_{1+u_j^0}, \quad (3.43)$$

where W is defined by

$$W \equiv \exp \left\{ i \int_0^1 dt \mathfrak{L}_\eta \right\}, \quad \eta_\mu(x) = i \partial_t U_t(x, \mu) U_t(x, \mu)^{-1}, \quad (3.44)$$

$Q_{\pm t}$ is defined by the evolution operator of the projector $P_{t\pm} = \hat{P}_{\pm} \Big|_{U=U_t}$ satisfying

$$\partial_t Q_{t\pm} = [\partial_t P_{t\pm}, P_{t\pm}] Q_{t\pm}, \quad Q_{0\pm} = 1, \quad (3.45)$$

and u_j^0, v_j^0 are the basis vectors for the reference link field at $t = 0$, $U_0^{(2)} \otimes U_0^{(1)}$. The basis is path-dependent and, in general, the fermion measure defined with this basis is also path-dependent. In fact, any two curves U_t and \tilde{U}_t define two different sets of the basis vectors, (v_j, u_j) and $(\tilde{v}_j, \tilde{u}_j)$, and the unitary transformation relating them does not necessarily has determinant 1. The fermion measure defined with the basis vectors is smooth if (and only if) it holds true for any closed curve U_t ($t \in [0, 1]; U_1 = U_0$) in the space $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ that

$$W = \det(1 - P_{0-} + P_{0-} Q_{1-}) \det(1 - P_{0+} + P_{0+} Q_{1+}). \quad (3.46)$$

This condition is referred as global integrability condition. The reconstruction theorem follows from the global integrability condition.

If a given closed curve is contractible, the global integrability condition reduces to eq. (3.36), the local version of the integrability condition. Then, what is actually required by the global integrability condition is that eq. (3.46) holds true for any non-contractible loops in the space $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$. Moreover, with the smooth deformation of a given non-contractible loop, the global integrability condition holds true. In particular, the base point (the point at $t = 0, 1$) of a non-contractible loop may be chosen arbitrarily in the given topological sector $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$. Then, one may choose $U_0 = U^{(2)} \otimes 1$ with a certain SU(2) link field in $\mathfrak{U}^{(2)}[Q]$ as the base point of non-contractible loops.

3.4.2 Non-contractible loops in the space of SU(2) × U(1) gauge fields

Since $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ is a direct product space, any non-contractible loop in $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ may be deformed to the product of the loops in $\mathfrak{U}^{(2)}[Q]$ and $\mathfrak{U}^{(1)}[0]$, respectively. Namely, one may assume that a non-contractible loop in $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ has the following form (without loss of generality):

$$U_t = \begin{cases} U_t^{(2)} \otimes 1 & (0 \leq t \leq 1; U_1^{(2)} = U_0^{(2)} = U^{(2)}), \\ U^{(2)} \otimes U_t^{(1)} & (1 \leq t \leq 2; U_1^{(1)} = U_2^{(1)} = 1), \end{cases} \quad (3.47)$$

with a certain SU(2) link field $U^{(2)}$ in $\mathfrak{U}^{(2)}[Q]$. Then, in order to prove the global integrability condition, one may consider separately the following two cases, (1) *non-contractible loops in $\mathfrak{U}^{(2)}[Q]$ with the trivial U(1) link field as a background* and (2) *non-contractible loops in $\mathfrak{U}^{(1)}[0]$ with an arbitrarily chosen SU(2) link field in $\mathfrak{U}^{(2)}[Q]$ as a background*.

In order to identify non-contractible loops in the topological sectors $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$, one needs to clarify the topological structure of the space of the admissible SU(2) × U(1) gauge fields.

As to the admissible U(1) gauge fields, it has been shown in [18] that the topological structure of $\mathfrak{U}^{(1)}[m]$ is a $(4 + L^4 - 1)$ -dimensional torus times a contractible space. Any admissible U(1) gauge field in a given topological sector $\mathfrak{U}^{(1)}[m]$ can be expressed as

$$U^{(1)}(x, \mu) = \tilde{U}^{(1)}(x, \mu) V_{[m]}(x, \mu), \quad (3.48)$$

where

$$V_{[m]}(x, \mu) = e^{-\frac{2\pi i}{L^2} [L\delta_{\hat{x}\mu, L-1} \sum_{\nu > \mu} m_{\mu\nu} \hat{x}_\nu + \sum_{\nu < \mu} m_{\mu\nu} \hat{x}_\nu]}. \quad (3.49)$$

Here the abbreviation $\hat{x}_\mu = x_\mu \bmod L$ has been used. $V_{[m]}(x, \mu)$ has the constant field tensor equal to $2\pi m_{\mu\nu}/L^2 (< \epsilon_1)$ and may be regarded as a reference field of $\mathfrak{U}^{(1)}[m]$. Then $\tilde{U}^{(1)}(x, \mu)$ stands for the dynamical degrees of freedom in the given topological sector. It can be parametrized with the three degrees of freedom:

$$\tilde{U}^{(1)}(x, \mu) = \Lambda(x) e^{iA_\mu^T(x)} U_{[w]}(x, \mu) \Lambda(x + \hat{\mu})^{-1}, \quad (3.50)$$

where $A_\mu^T(x)$ is the transverse vector potential satisfying

$$\partial_\mu^* A_\mu^T(x) = 0, \quad \sum_{x \in \Gamma} A_\mu^T(x) = 0, \quad (3.51)$$

$$\partial_\mu A_\nu^T(x) - \partial_\nu A_\mu^T(x) + 2\pi m_{\mu\nu}/L^2 = F_{\mu\nu}(x). \quad (3.52)$$

$U_{[w]}(x, \mu)$ represents the degrees of freedom of the Wilson lines defined by

$$U_{[w]}(x, \mu) = \begin{cases} w_\mu = \prod_{s=0}^{L-1} \{\tilde{U}^{(1)}(0 + s\hat{\mu}, \mu) e^{-iA_\mu^T(0+s\hat{\mu})}\} & \text{if } x_\mu = L-1, \\ 1 & \text{otherwise,} \end{cases} \quad (3.53)$$

and $\Lambda(x)$ is the gauge function satisfying $\Lambda(0) = 1$. By this parametrization, one can see that the space of the vector potentials $A_\mu^T(x)$, denoted by \mathfrak{A} , is contractible, while the space of the gauge functions $\Lambda(x)$, denoted by \mathfrak{G}_0 , is $(L^4 - 1)$ -dimensional torus. Therefore, the topological structure of $\mathfrak{U}^{(1)}[m]$ is a $(4 + L^4 - 1)$ -dimensional torus times a contractible space:

$$\mathfrak{U}^{(1)}[m] \simeq \text{U}(1)^4 \times \mathfrak{G}_0 \times \mathfrak{A}. \quad (3.54)$$

Then, one can see that there exist two kinds of non-contractible loops ($0 \leq t \leq 1$) in $\mathfrak{U}^{(1)}[m]$. The first one is the gauge loops given by

$$U_t^{(1)}(x, \mu) = \Lambda_t(x) V_{[m]}(x, \mu) \Lambda_t(x + \hat{\mu})^{-1}, \quad \Lambda_t(x) = \exp\{i2\pi t \delta_{\hat{x}\hat{y}}\}. \quad (3.55)$$

The second one is the non-gauge loops given by

$$U_t^{(1)}(x, \mu) = V_{[m]}(x, \mu) \exp\{i2\pi t \delta_{\mu\nu} \delta_{\hat{x}0}\}. \quad (3.56)$$

On the other hand, the topological structure of the space of the admissible SU(2) gauge fields, $\mathfrak{U}^{(2)}[Q]$, is not known so far [74]. But, as long as one considers only the topological sectors with vanishing U(1) magnetic flux, $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$, one can establish the global integrability condition without the knowledge, by virtue of the pseudo reality of SU(2).

3.4.3 SU(2) loops — use of the pseudo reality of SU(2) —

We first consider the case (1) *non-contractible loops in $\mathfrak{U}^{(2)}[Q]$ with the trivial U(1) link field as a background*. When $U^{(1)}(x, \mu) = 1$, only the SU(2) gauge field couples to the left-handed fermion $\psi_-(x)$, which now consists of four degenerate SU(2) doublets. By noting the

pseudo reality of SU(2), and the charge- and γ_5 -conjugation properties of the lattice Dirac operator, one may choose the basis vectors of the left-handed fermion $\psi_-(x) = (q_-^1(x), q_-^2(x), q_-^3(x), l_-(x))$ for any given SU(2) gauge field $U^{(2)}(x, \mu) \in \mathfrak{U}^{(2)}[Q]$ as given by eqs. (3.18), (3.19), (3.20) and (3.21). With this choice of the basis, one can infer that the measure term vanishes identically.

For any closed curve in the space $\mathfrak{U}^{(2)}[Q]$, $U_t^{(2)}(x, \mu)$ ($t \in [0, 1]$), one then has

$$W = 1. \tag{3.57}$$

On the other hand, from the hermiticity of P_{t-} , the unitarity of Q_{t-} and the charge conjugation properties of P_{t-} and Q_{t-} , it follows that

$$P_{t-} = (\gamma_5 C^{-1} \otimes i\sigma_2) \{P_{t-}\}^T (C\gamma_5 \otimes (i\sigma_2)^{-1}), \tag{3.58}$$

$$Q_{t-} = (\gamma_5 C^{-1} \otimes i\sigma_2) \{Q_{t-}^{-1}\}^T (C\gamma_5 \otimes (i\sigma_2)^{-1}). \tag{3.59}$$

Then one can infer that

$$\det(1 - P_{0-} + P_{0-}Q_{1-}) = \det(1 - P_{0-} + P_{0-}\{Q_{1-}\}^{-1}), \tag{3.60}$$

or

$$\det(1 - P_{0-} + P_{0-}Q_{1-}) = \pm 1. \tag{3.61}$$

Since $\psi_-(x)$ consists of four degenerate SU(2) doublets, P_{t-} and Q_{t-} factorize as

$$P_{t-} = \prod_{i=1}^4 \otimes P_{t-}^{(i)}, \quad Q_{t-} = \prod_{i=1}^4 \otimes Q_{t-}^{(i)}, \tag{3.62}$$

where $P_{t-}^{(i)}$ and $Q_{t-}^{(i)}$ ($i = 1, 2, 3, 4$) are the projection- and the evolution-operators for the i -th component SU(2) doublet, respectively, and each set of the operators $P_{t-}^{(i)}$ and $Q_{t-}^{(i)}$ satisfies the same identity as eq. (3.61). Therefore one obtains¹³

$$\begin{aligned} \det(1 - P_{0-} + P_{0-}Q_{1-}) &= \prod_{i=1}^4 \det \left(1 - P_{0-}^{(i)} + P_{0-}^{(i)}Q_{1-}^{(i)} \right) \\ &= \left[\det \left(1 - P_{0-}^{(1)} + P_{0-}^{(1)}Q_{1-}^{(1)} \right) \right]^4 \\ &= 1. \end{aligned} \tag{3.63}$$

Thus the global integrability condition holds true for any closed curves in $\mathfrak{U}^{(2)}[Q]$ with the trivial U(1) link field as a background.

The measure of the chiral fermion $\psi_-(x)$ can be defined globally within $\mathfrak{U}^{(2)}[Q]$ and the lattice counterpart of the SU(2) global anomaly [75–77] is absent in this case.

¹³The right-handed fermion does not contribute the integrability condition in this case: $\det(1 - P_{0+} + P_{0+}Q_{1+}) = 1$.

3.4.4 U(1) loops with SU(2) background

We next consider the case (2) *non-contactible loops in $\mathfrak{U}^{(1)}[0]$ with an arbitrarily chosen SU(2) link field in $\mathfrak{U}^{(2)}[Q]$ as a background.*

For gauge loops, one has

$$\mathfrak{L}_\eta = \text{tr}\{\gamma_5(1 - aD)(y, y)\}_{|_{U^{(2)} \otimes U_t^{(1)}}, \quad \eta_\mu(x) = -i U_t(x, \mu)^{-1} \partial_t U_t(x, \mu) = -\partial \delta_{\bar{x}\bar{y}}, \quad (3.64)$$

where the SU(2) gauge field $U^{(2)}(x, \mu)$ is chosen arbitrarily in $\mathfrak{U}^{(2)}[Q]$ and is fixed as a background. Then the l.h.s. is evaluated as

$$W = \exp\{i2\pi[\text{tr}\{Y_- \gamma_5(1 - D_L)(y, y)\} - \text{tr}\{Y_+ \gamma_5(1 - D_L)(y, y)\}]\}_{|_{t=0}}. \quad (3.65)$$

On the other hand, the factors in the r.h.s. are evaluated as

$$\begin{aligned} \det\{1 - P_{0\pm} + P_{0\pm} Q_{1\pm}\} &= \lim_{n \rightarrow \infty} \det\{1 - P_{0\pm} + (P_{0\pm} \Lambda_{\Delta t}^{-1} P_{0\pm})^n\} \\ &= \exp\{-i2\pi \text{Tr}[\omega Y_\pm P_{0\pm}]\}, \end{aligned} \quad (3.66)$$

where $\Delta t = 2\pi/n$ and $\omega(x) = \delta_{\bar{x}\bar{y}}$ and therefore

$$\begin{aligned} \det\{1 - P_{0-} + P_{0-} Q_{1-}\} \det\{1 - P_{0+} + P_{0+} Q_{1+}\} \\ = \exp\{-i2\pi \text{Tr}[\omega Y_- P_{0-}]\} \exp\{-i2\pi \text{Tr}[\omega Y_+ P_{0+}]\} = W. \end{aligned} \quad (3.67)$$

Thus the global integrability condition holds true for the gauge loops.

For non-gauge loops, one has

$$\mathfrak{L}_\eta = 2\pi j_\nu(0)|_{U^{(2)} \otimes U_t^{(1)}}, \quad \eta_\mu(x)_{(\nu)} = -i U_{[w]}(x, \mu)^{-1} \partial_{t_\nu} U_{[w]}(x, \mu) = 2\pi \delta_{\mu\nu} \delta_{\bar{x}0}, \quad (3.68)$$

where again the SU(2) gauge field $U^{(2)}(x, \mu)$ is chosen arbitrarily in $\mathfrak{U}^{(2)}[Q]$ and is fixed as a background. Noting the charge conjugation properties of the U(1) measure term current under the transformation, $U_{[w]} \rightarrow U_{[w]}^*$, $U^{(2)} \rightarrow U^{(2)*} = (i\sigma_2) U^{(2)} (i\sigma_2)^{-1}$:

$$j_\mu(x)|_{U^{(1)*}, U^{(2)}} = +j_\mu(x)|_{U^{(1)}, U^{(2)}}, \quad (3.69)$$

the l.h.s. can be evaluated as

$$W = \exp\left\{i \int_0^{2\pi} dt j_\nu(0)\right\} = \exp\left\{i \int_0^\pi dt j_\nu(0) - i \int_0^{-\pi} dt j_\nu(0)\right\} = 1 \quad (3.70)$$

On the other hand, the r.h.s. can be evaluated as ($n = 2r$)

$$\begin{aligned} \det\{1 - P_{0\pm} + P_{0\pm} Q_{1\pm}\} &= \lim_{n \rightarrow \infty} \det\{1 - P_{0\pm} + P_{0\pm} (\mathcal{C}_\pm)^{-1} P_{t_1\pm} \mathcal{C}_\pm \cdots (\mathcal{C}_\pm)^{-1} P_{t_r\pm} \mathcal{C}_\pm \\ &\quad \times P_{t_{r-1}\pm} P_{t_{r-2}\pm} \cdots P_{t_1\pm} P_{t_0\pm}\} \\ &= \det\{1 - P_{0\pm} + P_{0\pm} (\mathcal{C}_\pm)^{-1} P_{0\pm}\} \det\{1 - P_{\pi\pm} + P_{\pi\pm} (\mathcal{C}_\pm)^{-1} P_{\pi\pm}\} \end{aligned}$$

where $\mathcal{C}_+ = (\gamma_5 C^{-1})$ and $\mathcal{C}_- = (\gamma_5 C^{-1} \otimes i\sigma_2)$. Each factor in the final expression is ± 1 because $\{\mathcal{C}_\pm\}^2 = 1$. The total expression is unity because, for the case of the right-handed factor, all SU(2) singlets have even hyper-charges and, for the left-handed factor, all four SU(2) doublets have odd hyper-charges. Thus the global integrability condition holds true for the non-gauge loops, too.

This completes the proof of the global integrability condition, and therefore, the reconstruction theorem.

4. An explicit construction of the mesure term

In this section, we explicitly construct the local currents $j_\mu^a(x)$ ($a = 1, 2, 3$) and $j_\mu(x)$ which satisfy all the required properties for the reconstruction theorem in the topological sectors $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ with vanishing magnetic fluxes $m_{\mu\nu} = 0$. We follow the approach in our previous work for the $U(1)$ case [35], extending the construction there to the case of the $SU(2) \times U(1)$ chiral gauge theory.

4.1 Parametrization of $U(1)$ link fields and their variations in finite volume

We first discuss the parametrization of the link fields in finite volume and their variations. We adopt the parametrization of the $U(1)$ link fields given by eqs. (3.48) and (3.50). It is unique and the each factors, $A_\mu^T(x)$, $U_{[w]}(x, \mu)$ and $\Lambda(x)$, may be regarded as the smooth functionals of the original link field $U^{(1)}(x, \mu)$.

Accordingly, the variation of the $U(1)$ link field,

$$\delta_\eta U^{(1)}(x, \mu) = i \eta_\mu(x) U^{(1)}(x, \mu), \quad (4.1)$$

may be decomposed as follows:

$$\eta_\mu(x) = \eta_\mu^T(x) + \eta_{\mu[w]}(x) + \eta_\mu^\Lambda(x). \quad (4.2)$$

$\eta_\mu^T(x)$ is the transverse part of $\eta_\mu(x)$ defined by

$$\partial_\mu^* \eta_\mu^T(x) = 0, \quad \sum_{x \in \Gamma} \eta_\mu^T(x) = 0, \quad (4.3)$$

which may be given explicitly as

$$\eta_\mu^T(x) = \sum_{y \in \Gamma} G_L(x - y) \partial_\lambda^* (\partial_\lambda \eta_\mu(x) - \partial_\mu \eta_\lambda(x)). \quad (4.4)$$

$\eta_{\mu[w]}(x)$ is the variation along the Wilson lines defined by

$$\eta_{\mu[w]}(x) = \sum_\nu \eta_{(\nu)} \delta_{\mu\nu} \delta_{x_\nu, L-1}, \quad \eta_{(\nu)} = L^{-3} \sum_{y \in \Gamma} \eta_\nu(y). \quad (4.5)$$

$\eta_\mu^\Lambda(x)$ is the variation of the gauge degrees of freedom in the form,

$$\eta_\mu^\Lambda(x) = -\partial_\mu \omega_\eta(x), \quad \omega_\eta(0) = 0. \quad (4.6)$$

This decomposition is also unique by the following reason: for an arbitrary periodic vector field $\eta_\mu(x)$, the vector field defined by $a_\mu(x) = \eta_\mu(x) - \eta_\mu^T(x) - \eta_{\mu[w]}(x)$ has the vanishing field tensor $\partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) = 0$ and the vanishing wilson lines $\sum_{s=0}^{L-1} a_\mu(x + s\hat{\mu}) = 0$. Then, the sum $\omega_\eta(x)$ of the vector field $a_\mu(x)$ along any lattice path from x to the origin $x = 0$ is independent of the chosen path, periodic in x and $\omega_\eta(0) = 0$. It gives the gauge function which reproduces $a_\mu(x)$ in the pure gauge form, $a_\mu(x) = -\partial_\mu \omega_\eta(x)$. This proves the uniqueness of the decomposition. The action of the differential operator δ_η to each factors, $A_\mu^T(x)$, $U_{[w]}(x, \mu)$ and $\Lambda(x)$, is then given as follows:

$$\delta_\eta A_\mu^T(x) = \eta_\mu^T(x), \quad (4.7)$$

$$\delta_\eta U_{[w]}(x, \mu) = i \eta_{\mu[w]}(x) U_{[w]}(x, \mu), \quad (4.8)$$

$$\delta_\eta \Lambda(x) = i \omega_\eta(x) \Lambda(x). \quad (4.9)$$

4.2 A closed formula of the measure term in finite volume

We now give an explicit formula of the measure term for the admissible $SU(2) \otimes U(1)$ gauge fields in the topological sectors $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[0]$ with the vanishing magnetic fluxes $m_{\mu\nu} = 0$.¹⁴ For this purpose, we introduce a vector potential defined by

$$\tilde{A}'_{\mu}(x) = A_{\mu}^T(x) - \frac{1}{i} \partial_{\mu} [\ln \Lambda(x)]; \quad \frac{1}{i} \ln \Lambda(x) \in (-\pi, \pi], \quad (4.10)$$

and choose a one-parameter family of the gauge fields as

$$U_s(x, \mu) = U^{(2)}(x, \mu) \otimes \left[e^{is\tilde{A}'_{\mu}(x)} U_{[w]}(x, \mu) \right], \quad 0 \leq s \leq 1. \quad (4.11)$$

Then we consider the linear functional of the variational parameters $\eta_{\mu}^{(2)}(x)$ and $\eta_{\mu}^{(1)}(x)$ given by

$$\begin{aligned} \mathfrak{L}_{\eta}^{\diamond} = & i \int_0^1 ds \operatorname{Tr} \left\{ \hat{P}_- [\partial_s \hat{P}_-, \delta_{\eta} \hat{P}_-] \right\} + i \int_0^1 ds \operatorname{Tr} \left\{ \hat{P}_+ [\partial_s \hat{P}_+, \delta_{\eta} \hat{P}_+] \right\} \\ & + \delta_{\eta} \int_0^1 ds \sum_{x \in \Gamma_4} \left\{ \tilde{A}'_{\mu}(x) k_{\mu}(x) \right\} + \mathfrak{L}_{\eta}|_{U=U^{(2)} \otimes U_{[w]}}, \end{aligned} \quad (4.12)$$

where $k_{\mu}(x)$ is the gauge-invariant local current which satisfies $\partial_{\mu}^* k_{\mu}(x) = q_L^{(1)}(x)$ and transforms as an axial vector field under the lattice symmetries. The additional term $\mathfrak{L}_{\eta}|_{U=U^{(2)} \otimes U_{[w]}}$ is the measure term at the gauge fields $U_0(x, \mu) = U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$, which construction will be discussed in the following section 4.3.

The currents $j_{\mu}^{a \diamond}(x) (a = 1, 2, 3)$, $j_{\mu}^{\diamond}(x)$ defined by eq. (4.12),

$$\mathfrak{L}_{\eta}^{\diamond} = \sum_{x \in \Gamma} \{ \eta_{\mu}^a(x) j_{\mu}^{a \diamond}(x) + \eta_{\mu}(x) j_{\mu}^{\diamond}(x) \}, \quad (4.13)$$

may be regarded as a functional of the link variable $U(x, \mu)$ through the dependences on $U^{(2)}(x, \mu)$, $A_{\mu}^T(x)$, $\Lambda(x)$ ($\ln \Lambda(x)$) and $U_{[w]}(x, \mu)$. The action of the differential operator δ_{η} to the vector potential $\tilde{A}'_{\mu}(x)$ is evaluated as

$$\begin{aligned} \delta_{\eta} \tilde{A}'_{\mu}(x) &= \delta_{\eta} A_{\mu}^T(x) - \partial_{\mu} \left[\frac{1}{i} \{ \delta_{\eta} \Lambda(x) \} \Lambda(x)^{-1} \right] = \eta_{\mu}^T(x) - \partial_{\mu} \omega_{\eta}(x) \\ &= \eta_{\mu}(x) - \eta_{\mu[w]}(x), \end{aligned} \quad (4.14)$$

and the variation of $U_s(x, \mu)$ is given by

$$\begin{aligned} \delta_{\eta} U_s(x, \mu) &= i \eta_{\mu}^{(2)}(x) U^{(2)}(x, \mu) \otimes \left[e^{is\tilde{A}'_{\mu}(x)} U_{[w]}(x, \mu) \right] \\ &+ U^{(2)}(x, \mu) \otimes i \left\{ s(\eta_{\mu}^{(1)}(x) - \eta_{\mu[w]}(x)) + \eta_{\mu[w]}(x) \right\} \left[e^{is\tilde{A}'_{\mu}(x)} U_{[w]}(x, \mu) \right]. \end{aligned} \quad (4.15)$$

¹⁴A general strategy to construct the $SU(2)$ part of the measure term was discussed in [29]. We follow this strategy, specifying explicitly the $U(1)$ part of the measure term current $j_{\mu}(x)$ and the interpolation path in the $U(1)$ direction.

The linear functional $\mathfrak{L}_\eta^\diamond$ so obtained, however, does not respect the lattice symmetries. In order to make it to transform as a pseudo scalar field under the lattice symmetries, we should average it over the lattice symmetries with the appropriate weights so as to project to the pseudo scalar component. Namely, we take the average as follows:¹⁵

$$\bar{\mathfrak{L}}_\eta^\diamond = \frac{1}{2^4 4!} \sum_{R \in O(4, \mathbb{Z})} \det R \mathfrak{L}_\eta^\diamond|_{U \rightarrow \{U\}^{R^{-1}}, \eta_\mu \rightarrow \{\eta_\mu\}^{R^{-1}}}. \quad (4.16)$$

Our main result is then stated as follows:

Lemma 2. *The currents $j_\mu^{a\diamond}(x) (a = 1, 2, 3)$, $j_\mu^\diamond(x)$ defined by eq. (4.12),*

$$\mathfrak{L}_\eta^\diamond = \sum_{x \in \Gamma} \{ \eta_\mu^a(x) j_\mu^{a\diamond}(x) + \eta_\mu(x) j_\mu^\diamond(x) \},$$

fulfills all the properties required for the reconstruction theorem in the lattice Glashow-Weinberg-Salam model except the transformation property under the lattice symmetries. It may be corrected by invoking the average eq. (4.16) over the lattice symmetries with the appropriate weights so as to project to the pseudo scalar component.

The proof of this statement will be given in section 4.4. The locality property of the currents will be examined in section 4.5.

4.3 Measure term at $U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$

The measure term at the gauge fields $U(x, \mu) = U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$ should consist of the two components:

$$\mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}} = \begin{cases} \mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}; \eta = \eta_{[w]}} & \text{for } \eta_\mu(x) = \eta_\mu^{(1)}(x) = \eta_{\mu[w]}(x), \\ \mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}; \eta = \eta^{(2)}} & \text{for } \eta_\mu(x) = \eta_\mu^{(2)}(x). \end{cases} \quad (4.17)$$

In order to construct the measure term at the gauge field $U(x, \mu) = U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$ with the variational parameters in the directions of the U(1) Wilson lines $\eta_{\mu[w]}(x)$, we first discuss a special property of the curvature terms associated with the U(1) Wilson lines $U_{[w]}(x, \mu)$, which turn out to be useful in the construction of a solution to the integrability condition eq. (3.36). Let us parametrize the Wilson lines $U_{[w]}(x, \mu)$ defined by eq. (3.53) as

$$w_\nu = \exp(it_\nu), \quad t_\nu \in [0, 2\pi) \quad (\nu = 1, 2, 3, 4), \quad (4.18)$$

and the variational parameters in the directions of the Wilson lines as

$$\lambda_{\mu(\nu)}(x) = \frac{1}{i} \partial_{t_\nu} U_{[w]}(x, \mu) \cdot U_{[w]}(x, \mu)^{-1} = \delta_{\mu\nu} \delta_{x_\nu, L-1}. \quad (4.19)$$

¹⁵In doing the average, one should note the fact that under the lattice symmetries the Wilson lines $U_{[w]}(x, \mu)$ are transformed to other Wilson lines $U_{[w']}(x, \mu)$ modulo gauge transformations, $\{U_{[w]}(x, \mu)\}^{R^{-1}} = U_{[w']}(x, \mu) \Lambda(x) \Lambda(x + \hat{\mu})^{-1}$. Accordingly, the variational parameter $\eta_{\mu[w]}(x)$ is transformed as $\{\eta_{\mu[w]}(x)\}^{R^{-1}} = \eta_{\mu[w']}(x) - \partial_\mu \omega(x)$ with a certain periodic gauge function $\omega(x)$.

The curvature term for the Wilson lines reads

$$\left[i\text{Tr}\{\hat{P}_+[\partial_{t_\mu}\hat{P}_+, \partial_{t_\nu}\hat{P}_+]\} + i\text{Tr}\{\hat{P}_-[\partial_{t_\mu}\hat{P}_-, \partial_{t_\nu}\hat{P}_-]\} \right]_{U=U^{(2)}\otimes U_{[w]}V_{[m]}} \equiv \mathfrak{C}_{\mu\nu}(t), \quad (4.20)$$

where $t = (t_1, t_2, t_3, t_4)$. Then the following lemma holds true:

Lemma 3. *In the topological sectors $\mathfrak{U}^{(2)}[Q] \otimes \mathfrak{U}^{(1)}[m]$ of the lattice Glashow-Weinberg-Salam model, the curvature term for the U(1) Wilson lines $\mathfrak{C}_{\mu\nu}(t)$, which possesses the properties*

$$\mathfrak{C}_{\mu\nu}(t) = -\mathfrak{C}_{\nu\mu}(t), \quad \partial_\mu\mathfrak{C}_{\nu\lambda}(t) + \partial_\nu\mathfrak{C}_{\lambda\mu}(t) + \partial_\lambda\mathfrak{C}_{\mu\nu}(t) = 0, \quad (4.21)$$

satisfies the bound

$$|\mathfrak{C}_{\mu\nu}(t)| \leq \kappa L^\sigma e^{-L/\varrho} \quad (4.22)$$

for certain positive constants κ and σ , while ϱ is the localization range of the lattice Dirac operator D . For a sufficiently large volume L^4 , it then follows that

$$\int_0^{2\pi} dt_\mu \int_0^{2\pi} dt_\nu \mathfrak{C}_{\mu\nu}(t) = 0, \quad (4.23)$$

and there exists smooth periodic vector field $\mathfrak{W}_\mu(t)$ such that

$$\mathfrak{C}_{\mu\nu}(t) = \partial_\mu\mathfrak{W}_\nu(t) - \partial_\nu\mathfrak{W}_\mu(t), \quad |\mathfrak{W}_\mu(t)| \leq 3\pi \sup_{t,\mu,\nu} |\mathfrak{C}_{\mu\nu}(t)|. \quad (4.24)$$

The proof of this lemma has been given for the U(1) case in [35], which holds true also for the SU(2) \otimes U(1) case here by regarding the SU(2) gauge field in the background. The proof is based on the fact that in infinite-volume the periodic link field which represents the degrees of freedom of the Wilson lines can be written in the pure-gauge form,

$$U_{[w]}(x, \mu) = \Lambda_{[w]}(x)\Lambda_{[w]}(x + \hat{\mu})^{-1}, \quad \Lambda_{[w]}(x) = \prod_{\mu} (w_\mu)^{n_\mu} \quad \text{for } x - nL \in \Gamma, \quad (4.25)$$

and therefore the gauge-invariant function of the link field in infinite volume is actually independent of the degrees of freedom of the Wilson lines. In fact, noting the representation eq. (2.11), one may rewrite the curvature term eq. (3.39) into

$$\mathfrak{C}_{\eta\zeta} = i\text{Tr} \left\{ Q_\Gamma \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + i\text{Tr} \left\{ Q_\Gamma \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} + \mathfrak{R}_{\eta\zeta}, \quad (4.26)$$

where Q_Γ here is the projector acting on the fields in infinite volume as

$$Q_\Gamma \psi(x) = \begin{cases} \psi(x) & \text{if } x \in \Gamma, \\ 0 & \text{otherwise.} \end{cases} \quad (4.27)$$

$\mathfrak{R}_{\eta\zeta}$ is the finite-volume correction to the curvature term given by

$$\begin{aligned} \mathfrak{R}_{\eta\zeta} = & i \sum_{s=\mp} \sum_{x \in \Gamma} \sum_{y, z \in \mathbb{Z}^4} \sum_{n \in \mathbb{Z}^4, n \neq 0} \text{tr} \{ P_s(x, y) \\ & \times [\delta_\eta P_s(y, z) \delta_\zeta P_s(z, x + Ln) - \delta_\zeta P_s(y, z) \delta_\eta P_s(z, x + Ln)] \}, \end{aligned} \quad (4.28)$$

while $P_s(x, y)(s = \mp)$ are the kernels of the chiral projectors in infinite volume,

$$P_{\mp}(x, y) = \frac{1}{2}(1 \mp \gamma_5)\delta_{xy} \pm \frac{1}{2}\gamma_5 D(x, y). \quad (4.29)$$

From eq. (2.12), one can infer that

$$|\mathfrak{R}_{\eta\zeta}| \leq \kappa_1 L^{\nu_1} e^{-L/\varrho} \|\eta\|_{\infty} \|\zeta\|_{\infty} \quad (4.30)$$

for some constants $\kappa_1 > 0$ and $\nu_1 \geq 0$. We then recall the fact that there exists the measure term in infinite volume [28], $\mathfrak{R}_{\eta} = \sum_{x \in \Gamma} \{\eta_{\mu}^a(x) j_{\mu}^{*a}(x) + \eta_{\mu}(x) j_{\mu}^*(x)\}$, which satisfies the integrability condition

$$i\text{Tr} \left\{ Q_{\Gamma} \hat{P}_{-} [\delta_{\eta} \hat{P}_{-}, \delta_{\zeta} \hat{P}_{-}] \right\} + i\text{Tr} \left\{ Q_{\Gamma} \hat{P}_{+} [\delta_{\eta} \hat{P}_{+}, \delta_{\zeta} \hat{P}_{+}] \right\} = \delta_{\eta} \mathfrak{R}_{\zeta} - \delta_{\zeta} \mathfrak{R}_{\eta} + \mathfrak{R}_{[\eta, \zeta]}. \quad (4.31)$$

The currents $j_{\mu}^{*a}(x)$ and $j_{\mu}^*(x)$ are defined for all admissible gauge fields in infinite volume and it is local and gauge-invariant under the U(1) gauge transformations. ($j_{\mu}^{*a}(x)$ and $j_{\mu}^*(x)$ are gauge-covariant and gauge-invariant, respectively, under the SU(2) gauge transformation.) Then, as discussed above, the currents are actually independent of the Wilson lines and the curvature of \mathfrak{R}_{η} evaluated in the directions of the Wilson lines vanishes identically. Namely,

$$\begin{aligned} & \left[\delta_{\lambda(\mu)} \mathfrak{R}_{\lambda(\nu)} - \delta_{\lambda(\nu)} \mathfrak{R}_{\lambda(\mu)} \right]_{U=U^{(2)} \otimes U_{[w]} V_{[m]}} \\ &= i\text{Tr} \left\{ Q_{\Gamma} \hat{P}_{-} [\delta_{\lambda(\mu)} \hat{P}_{-}, \delta_{\lambda(\nu)} \hat{P}_{-}] \right\} + i\text{Tr} \left\{ Q_{\Gamma} \hat{P}_{+} [\delta_{\lambda(\mu)} \hat{P}_{+}, \delta_{\lambda(\nu)} \hat{P}_{+}] \right\} \Big|_{U=U^{(2)} \otimes U_{[w]} V_{[m]}} \\ &= \mathfrak{C}_{\mu\nu}(t) - \mathfrak{R}_{\lambda(\mu)\lambda(\nu)} = 0. \end{aligned} \quad (4.32)$$

This fact immediately implies that the curvature term for the Wilson lines, $\mathfrak{C}_{\mu\nu}$, itself satisfies the bound eq. (4.22) and because of this bound, the two-dimensional integration of the curvature, which should be a multiple of 2π , must vanish identically for a sufficiently large L . The existence of the smooth periodic vector field $\mathfrak{W}_{\mu}(t)$ then follows from the lemma 9.2 in [18].

By the above lemma, one can construct a solution of the integrability condition at the gauge fields $U(x, \mu) = U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$,

$$\delta_{\lambda(\mu)} \mathfrak{W}_{\nu} - \delta_{\lambda(\nu)} \mathfrak{W}_{\mu} = \mathfrak{C}_{\mu\nu} \Big|_{U=U^{(2)} \otimes U_{[w]}}, \quad (4.33)$$

from $\mathfrak{C}_{\mu\nu}$ directly. The solution may be given explicitly by the formulae,

$$\begin{aligned} \mathfrak{W}_4 &= \frac{1}{2\pi} \int_0^{2\pi} dr_4 \int_0^{(t_1, t_2, t_3)} \{ dr_1 \mathfrak{C}_{14} + dr_2 \mathfrak{C}_{24} + dr_3 \mathfrak{C}_{34} \}, \\ \mathfrak{W}_3 &= \int_0^{t_4} dr_4 \mathfrak{C}_{43} - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 \mathfrak{C}_{43} + \left[\frac{1}{2\pi} \int_0^{2\pi} dr_3 \int_0^{(t_1, t_2)} \{ dr_1 \mathfrak{C}_{13} + dr_2 \mathfrak{C}_{23} \} \right]_{t_4=0}, \\ \mathfrak{W}_2 &= \int_0^{t_4} dr_4 \mathfrak{C}_{42} - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 \mathfrak{C}_{42} \\ &+ \left[\int_0^{t_3} dr_3 \mathfrak{C}_{32} - \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 \mathfrak{C}_{32} \right]_{t_4=0} + \left[\frac{1}{2\pi} \int_0^{2\pi} dr_2 \int_0^{(t_1)} \{ dr_1 \mathfrak{C}_{12} \} \right]_{t_4=t_3=0}, \end{aligned}$$

$$\begin{aligned} \mathfrak{W}_1 = & \int_0^{t_4} dr_4 \mathfrak{C}_{41} - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 \mathfrak{C}_{41} \\ & + \left[\int_0^{t_3} dr_3 \mathfrak{C}_{31} - \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 \mathfrak{C}_{31} \right]_{t_4=0} + \left[\int_0^{t_2} dr_2 \mathfrak{C}_{21} - \frac{t_2}{2\pi} \int_0^{2\pi} dr_2 \mathfrak{C}_{21} \right]_{t_4=t_3=0}. \end{aligned} \quad (4.34)$$

It follows from the properties of $\mathfrak{C}_{\mu\nu}$ that this solution is periodic and smooth with respect to the Wilson lines $U_{[w]}$ and satisfies the bound

$$|\mathfrak{W}_\mu| \leq \kappa_2 L^{\nu_1} e^{-L/\varrho}, \quad (4.35)$$

for certain positive constants κ_2 and ν_2 . It also follows that this solution is gauge invariant. Then the linear functional of the variational parameters $\eta_{\mu[w]}(x)$, $\sum_\nu \eta_{(\nu)} \mathfrak{W}_\nu$, provides the measure term at the gauge field $U(x, \mu) = U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu) V_{[m]}(x, \mu)$ with the variational parameters in the directions of the U(1) Wilson lines $\eta_{\mu[w]}(x)$:

$$\mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}; \eta=\eta_{[w]}} = \sum_\nu \eta_{(\nu)} \mathfrak{W}_\nu. \quad (4.36)$$

On the other hand, the measure term at the gauge field $U(x, \mu) = U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$ with the variational parameters in the directions of the SU(2) gauge fields may be given by the following formulae:

$$\begin{aligned} \mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}; \eta=\eta^{(2)}} = & \int_0^{t_1} dr_1 \mathfrak{C}_{1\eta}(r_1, 0, 0, 0) + \int_0^{t_2} dr_2 \mathfrak{C}_{2\eta}(t_1, r_2, 0, 0) \\ & + \int_0^{t_3} dr_3 \mathfrak{C}_{3\eta}(t_1, t_2, r_3, 0) + \int_0^{t_4} dr_4 \mathfrak{C}_{4\eta}(t_1, t_2, t_3, r_4) - \delta_\eta \phi_{[w]}, \end{aligned} \quad (4.37)$$

and

$$\begin{aligned} \phi_{[w]} = & \int_0^{(t_1)} dr_1 \mathfrak{W}_1(r_1, 0, 0, 0) + \int_0^{(t_2)} dr_2 \mathfrak{W}_2(t_1, r_2, 0, 0) \\ & + \int_0^{(t_3)} dr_3 \mathfrak{W}_3(t_1, t_2, r_3, 0) + \int_0^{(t_4)} dr_4 \mathfrak{W}_4(t_1, t_2, t_3, r_4). \end{aligned} \quad (4.38)$$

It is not difficult to show that the measure term so constructed indeed satisfies the integrability condition for all possible directions of the variational parameters. It also follows from the properties of $\mathfrak{C}_{\mu\nu}$ and $\mathfrak{C}_{\mu\eta^{(2)}}$ that the measure term current is gauge covariant and is periodic and smooth with respect to the Wilson lines $U_{[w]}(x, \mu)$. To see the latter property explicitly, one may rewrite the above formula as follows:

$$\begin{aligned} \mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}; \eta=\eta^{(2)}} & = \int_0^{t_1} dr_1 \mathfrak{C}_{1\eta}(r_1, 0, 0, 0) \\ & + \int_0^{t_2} dr_2 \mathfrak{C}_{2\eta}(t_1, r_2, 0, 0) - \frac{t_2}{2\pi} \int_0^{2\pi} dr_2 \mathfrak{C}_{2\eta}(t_1, r_2, 0, 0) + \frac{t_2}{2\pi} \int_0^{2\pi} dr_2 \mathfrak{C}_{2\eta}(0, r_2, 0, 0) \\ & + \int_0^{t_3} dr_3 \mathfrak{C}_{3\eta}(t_1, t_2, r_3, 0) - \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 \mathfrak{C}_{3\eta}(t_1, t_2, r_3, 0) + \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 \mathfrak{C}_{3\eta}(0, 0, r_3, 0) \\ & + \int_0^{t_4} dr_4 \mathfrak{C}_{4\eta}(t_1, t_2, t_3, r_4) - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 \mathfrak{C}_{4\eta}(t_1, t_2, t_3, r_4) + \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 \mathfrak{C}_{4\eta}(0, 0, 0, r_4). \end{aligned}$$

Then, from the pseudo reality of $SU(2)$ and the charge conjugation property of $\mathfrak{C}_{\nu\eta}$ ($\nu = 1, 2, 3, 4$),

$$\mathfrak{C}_{\nu\eta}|_{U=U^{(2)} \otimes \{U_t^{(1)}\}^*} = \mathfrak{C}_{\nu\eta}|_{U=U^{(2)} \otimes U_t^{(1)}}, \quad (4.39)$$

one can infer that

$$\begin{aligned} \int_0^{2\pi} dr_\nu \mathfrak{C}_{\nu\eta}(0, \dots, r_\nu, \dots, 0) &= \int_0^\pi dr_\nu [\mathfrak{C}_{\nu\eta}(0, \dots, r_\nu, \dots, 0) - \mathfrak{C}_{\nu\eta}(0, \dots, -r_\nu, \dots, 0)] \\ &= 0 \quad (\nu = 1, 2, 3, 4). \end{aligned} \quad (4.40)$$

With these identities, one can easily verify that the measure term is periodic and smooth with respect to the Wilson lines $U_{[w]}(x, \mu)$.

Finally, we note that the measure term $\mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}}$ so constructed satisfies the bound

$$\left| \mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}} \right| \leq \kappa' L^{\sigma'} e^{-L/\varrho} \|\eta\|_\infty \quad (4.41)$$

for certain positive constants κ' and σ' . For $\eta_\mu(x) = \eta_{\mu[w]}(x)$, it immediately follows from eq. (4.35). For $\eta_\mu(x) = \eta_\mu^{(2)}(x)$, as one can see from the argument given in the proof of the lemma 3 and eqs. (4.25) and (4.31), the curvature term $\mathfrak{C}_{\eta\zeta} - \mathfrak{R}_{\eta\zeta}$ does not actually depend on the $U(1)$ Wilson lines. Then, one may write

$$\begin{aligned} \mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}; \eta=\eta^{(2)}} &= \int_0^{t_1} dr_1 \mathfrak{R}_{1\eta}(r_1, 0, 0, 0) \\ &+ \int_0^{t_2} dr_2 \mathfrak{R}_{2\eta}(t_1, r_2, 0, 0) - \frac{t_2}{2\pi} \int_0^{2\pi} dr_2 \mathfrak{R}_{2\eta}(t_1, r_2, 0, 0) \\ &+ \int_0^{t_3} dr_3 \mathfrak{R}_{3\eta}(t_1, t_2, r_3, 0) - \frac{t_3}{2\pi} \int_0^{2\pi} dr_3 \mathfrak{R}_{3\eta}(t_1, t_2, r_3, 0) \\ &+ \int_0^{t_4} dr_4 \mathfrak{R}_{4\eta}(t_1, t_2, t_3, r_4) - \frac{t_4}{2\pi} \int_0^{2\pi} dr_4 \mathfrak{R}_{4\eta}(t_1, t_2, t_3, r_4), \end{aligned}$$

and the bound follows from eq. (4.30).

4.4 Proof of the lemma 2

We give a proof that the local currents, $j_\mu^{\circ a}(x)$ and $j_\mu^\diamond(x)$, defined by eq. (4.12) satisfy all the properties required for the reconstruction theorem. Although the proof is quite similar to that of theorem 5.3 in [18], or that given in [35], we give it here for completeness.

1. *Smoothness.* By construction, $j_\mu^{\circ a}(x)$, $j_\mu^\diamond(x)$ are defined for all admissible gauge fields in $\mathfrak{U}_Q^{(2)} \otimes \mathfrak{U}_{[0]}^{(1)}$. It depends smoothly on the link fields $U^{(2)}(x, \mu)$, $\tilde{A}'_\mu(x)$ and $U_{[w]}(x, \mu)$ because \hat{P}_- and k_μ are smooth functions of $U_s(x, \mu)$. Although $\tilde{A}'_\mu(x)$ is not continuous when $\Lambda(x) = -1$ at some points x because of the cut in $\ln \Lambda(x)$, its discontinuity is always in the pure-gauge form

$$\text{disc.}\{\tilde{A}'_\mu(x)\} = -\partial_\mu \omega(x); \quad \omega(0) = 0, \quad (4.42)$$

where the gauge function $\omega(x)$ takes values that are integer multiples of 2π . Then, any smooth functionals of $\tilde{A}'_\mu(x)$ are smooth with respect to the link field $U^{(1)}(x, \mu)$,

if they are gauge-invariant under the gauge transformations $\tilde{A}'_\mu(x) \rightarrow \tilde{A}'_\mu(x) + \partial_\mu \omega(x)$ for arbitrary periodic gauge functions $\omega(x)$ satisfying $\omega(0) = 0$. The currents $j_\mu^{\circ a}(x)$ and $j_\mu^\diamond(x)$ are indeed gauge-invariant under such gauge transformations. Namely, taking the gauge covariance of $\hat{P}_-(x, y)$ and the gauge invariance of $k_\mu(x)$ into account, the change of $\mathfrak{L}_\eta^\diamond$ under the gauge transformations

$$\begin{aligned}
 & \int_0^1 ds \text{Tr} \{ \hat{P}_- [[\omega Y_-, \hat{P}_-], \delta_\eta \hat{P}_-] \} + \int_0^1 ds \text{Tr} \{ \hat{P}_+ [[\omega Y_+, \hat{P}_+], \delta_\eta \hat{P}_+] \} \\
 & + \int_0^1 ds \sum_{x \in \Gamma} \partial_\mu \omega(x) \delta_\eta \bar{k}_\mu(x) \\
 & = - \int_0^1 ds \text{Tr} \{ \omega Y_- \delta_\eta \hat{P}_- \} - \int_0^1 ds \text{Tr} \{ \omega Y_+ \delta_\eta \hat{P}_+ \} + \int_0^1 ds \sum_{x \in \Gamma} \partial_\mu \omega(x) \delta_\eta k_\mu(x) \\
 & = \int_0^1 ds \sum_{x \in \Gamma} \omega(x) \delta_\eta \{ -\text{tr} \{ Y_- \gamma_5 D \} (x, x) + \text{tr} \{ Y_+ \gamma_5 D \} (x, x) - \partial_\mu^* k_\mu(x) \} = 0, \quad (4.43)
 \end{aligned}$$

where the identity $\hat{P}_\pm \delta_\eta \hat{P}_\pm \hat{P}_\pm = 0$ has been used.

2. *Gauge invariance/covariance and symmetry properties.* The gauge invariance of $j_\mu^{\circ a}(x)$ and $j_\mu^\diamond(x)$ under the U(1) gauge transformations has been shown above. The transformation properties of $j_\mu^{\circ a}(x)$, $j_\mu^\diamond(x)$ under the SU(2) gauge transformations and the lattice symmetries are also evident from the transformation properties of \hat{P}_- , k_μ and $\mathfrak{L}_\eta|_{U=U^{(2)} \otimes U_{[w]}}$.
3. *Integrability condition.* From the definition of $\mathfrak{L}_\eta^\diamond$, eq. (4.12), one finds immediately that the second term does not contribute to the curvature $\delta_\eta \mathfrak{L}_\zeta^\diamond - \delta_\zeta \mathfrak{L}_\eta^\diamond + \mathfrak{L}_{[\eta, \zeta]}^\diamond$ and the third term gives the curvature term at the gauge fields, $U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$. Taking the identity $\text{Tr} \{ \delta_1 \hat{P}_\pm \delta_2 \hat{P}_\pm \delta_3 \hat{P}_\pm \} = 0$ into account, the curvature is evaluated as

$$\begin{aligned}
 \delta_\eta \mathfrak{L}_\zeta^\diamond - \delta_\zeta \mathfrak{L}_\eta^\diamond + \mathfrak{L}_{[\eta, \zeta]}^\diamond & = i \int_0^1 ds \text{Tr} \left\{ \hat{P}_- [\delta_\eta \partial_s \hat{P}_-, \delta_\zeta \hat{P}_-] - \hat{P}_- [\delta_\zeta \partial_s \hat{P}_-, \delta_\eta \hat{P}_-] \right\} \quad (4.44) \\
 & + i \int_0^1 ds \text{Tr} \left\{ \hat{P}_+ [\delta_\eta \partial_s \hat{P}_+, \delta_\zeta \hat{P}_+] - \hat{P}_+ [\delta_\zeta \partial_s \hat{P}_+, \delta_\eta \hat{P}_+] \right\} \\
 & + \left[i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + i \text{Tr} \left\{ \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} \right] \Big|_{U=U^{(2)} \otimes U_{[w]}} \\
 & = i \int_0^1 ds \partial_s \left[\text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + \text{Tr} \left\{ \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} \right] \\
 & + \left[i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + i \text{Tr} \left\{ \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} \right] \Big|_{U=U^{(2)} \otimes U_{[w]}}.
 \end{aligned}$$

After the integration in the first term, the contributions from the lower end of the integration range cancels with the second term, because the variational parameters for the U(1) gauge field in this contribution is restricted to $\eta_{\mu[w]}(x)$:

$$\delta_{\eta(1)} U_s(x, \mu) U_s(x, \mu)^{-1} \Big|_{s=0} = [s(\eta_\mu^{(1)}(x) - \eta_{\mu[w]}(x)) + \eta_{\mu[w]}(x)]_{s=0} = \eta_{\mu[w]}(x). \quad (4.45)$$

4. *Anomalous conservation law.* If one sets $\eta_\mu^{(1)}(x) = -\partial_\mu\omega(x)$ (where $\omega(x)$ is any lattice function on Γ), the left-hand side of eq. (4.12) becomes

$$\sum_{x \in \Gamma} \omega(x) \partial_\mu^* j_\mu^\diamond(x). \quad (4.46)$$

On the other hand, using the identities

$$\delta_\eta \hat{P}_\pm = is \left[\omega Y_\pm, \hat{P}_\pm \right], \quad \delta_\eta k_\mu(x) = 0, \quad (4.47)$$

the right-hand side is evaluated as

$$\begin{aligned} & - \int_0^1 ds s \operatorname{Tr} \{ \omega Y_+ \partial_s \hat{P}_+ \} - \int_0^1 ds s \operatorname{Tr} \{ \omega Y_- \partial_s \hat{P}_- \} - \int_0^1 ds \sum_{x \in \Gamma} \partial_\mu \omega(x) k_\mu(x) \\ & = \sum_{x \in \Gamma} \omega(x) \{ \operatorname{tr} \{ Y_+ \gamma_5 D_L \}(x, x) - \operatorname{tr} \{ Y_- \gamma_5 D_L \}(x, x) \} \\ & \quad + \int_0^1 ds \sum_{x \in \Gamma} \omega(x) \{ -\operatorname{tr} \{ Y_+ \gamma_5 D_L \}(x, x) + \operatorname{tr} \{ Y_+ \gamma_5 D_L \}(x, x) + \partial_\mu^* k_\mu(x) \} \\ & = \sum_{x \in \Gamma} \omega(x) \{ -\operatorname{tr} \{ Y_+ \gamma_5 (1 - D_L) \}(x, x) + \operatorname{tr} \{ Y_+ \gamma_5 (1 - D_L) \}(x, x) \}. \end{aligned} \quad (4.48)$$

Also, if one sets $\eta_\mu^{(2)}(x) = -\nabla_\mu \omega(x)$, the left-hand side of eq. (4.12) becomes

$$\sum_{x \in \Gamma} \omega^a(x) \{ \nabla_\mu^* j_\mu^\diamond \}^a(x). \quad (4.49)$$

On the other hand, using the identities

$$\delta_\eta \hat{P}_- = i \left[\omega, \hat{P}_- \right], \quad \delta_\eta k_\mu(x) = 0, \quad (4.50)$$

the right-hand side is evaluated as

$$\begin{aligned} & - \int_0^1 ds \partial_s \operatorname{Tr} \{ \omega \hat{P}_- \} + \mathfrak{L}_\eta |_{U=U^{(2)} \otimes U_{[w]}; \eta_\mu^{(2)} = -\nabla_\mu \omega} \\ & = - \sum_{x \in \Gamma} \omega^a(x) \operatorname{tr} \{ T^a \gamma_5 D_L \}(x, x) \\ & \quad + \sum_{x \in \Gamma} \omega^a(x) \{ \operatorname{tr} \{ T^a \gamma_5 D_L \}(x, x) + \{ \nabla_\mu j_\mu^\diamond \}^a(x) \} |_{U=U^{(2)} \otimes U_{[w]}}. \end{aligned} \quad (4.51)$$

The last term vanishes identically if the anomalous conservation laws hold for the measure term $\mathfrak{L}_\eta |_{U=U^{(2)} \otimes U_{[w]}}$ at the gauge fields $U^{(2)}(x, \mu) \otimes U_{[w]}(x, \mu)$. This follows from its definition eq. (4.37) by noting

$$\mathfrak{C}_{\nu\eta}(t) |_{\eta_\mu^{(2)} = -\nabla_\mu \omega} = -\partial_{t_\nu} \operatorname{Tr} \{ \omega \hat{P}_- \}(t), \quad \delta_\eta \phi_{[w]} |_{\eta_\mu^{(2)} = -\nabla_\mu \omega} = 0, \quad (4.52)$$

and the fact that the SU(2) gauge anomaly $\operatorname{tr} \{ T^a \gamma_5 (1 - D_L) \}(x, x)$ vanishes identically when the U(1) gauge field is trivial ($t = 0$) due to the pseudo reality of SU(2).

4.5 Locality properties of the measure term currents

Finally, we examine the locality property of the measure term currents, $j_\mu^{\diamond a}(x)$ and $j_\mu^\diamond(x)$. We follow the procedure to decompose the measure term eq. (4.12) into the part definable in infinite volume and the part of the finite volume corrections. Namely, the measure term eq. (4.12) may be decomposed as follows:

$$\mathfrak{L}_\eta^\diamond = \mathfrak{K}_\eta^\diamond + \mathfrak{S}_\eta^\diamond, \quad (4.53)$$

where

$$\begin{aligned} \mathfrak{K}_\eta^\diamond &= i \int_0^1 ds \operatorname{Tr} \left\{ Q_\Gamma \hat{P}_- [\partial_s \hat{P}_-, \delta_\eta \hat{P}_-] \right\} + i \int_0^1 ds \operatorname{Tr} \left\{ Q_\Gamma \hat{P}_+ [\partial_s \hat{P}_+, \delta_\eta \hat{P}_+] \right\} \\ &\quad + \delta_\eta \int_0^1 ds \sum_{x \in \Gamma} \left\{ \tilde{A}'_\mu(x) \bar{k}_\mu(x) \right\}, \end{aligned} \quad (4.54)$$

$$\begin{aligned} \mathfrak{S}_\eta^\diamond &= \int_0^1 ds \mathfrak{R}_{\zeta\eta} |_{\zeta_\mu = \tilde{A}'_\mu} \\ &\quad + \delta_\eta \int_0^1 ds \sum_{x \in \Gamma} \left\{ \tilde{A}'_\mu(x) \Delta k_\mu(x) \right\} + \mathfrak{L}_\eta |_{U=U^{(2)} \otimes U_{[w]}}. \end{aligned} \quad (4.55)$$

From the bounds eqs. (3.34), (4.30) and eq. (4.41) and $\|A_\mu^T(x)\| \leq \kappa_6 L^4 (\kappa_6 > 0)$ [18], one can infer

$$|\mathfrak{S}_\eta^\diamond| \leq \kappa_3 L^{\nu_3} e^{-L/\varrho} \|\eta\|_\infty \quad (4.56)$$

for some constants $\kappa_3 > 0$, $\nu_3 \geq 0$.

As to $\mathfrak{K}_\eta^\diamond$ defined by eq. (4.54), if one introduces the truncated fields

$$\eta_\mu^n(x) = \begin{cases} \eta_\mu(x) & \text{if } x - Ln \in \Gamma, \\ 0 & \text{otherwise,} \end{cases} \quad (4.57)$$

for any integer vector n , it may be rewritten into

$$\begin{aligned} \mathfrak{K}_\eta^\diamond &= i \int_0^1 ds \operatorname{Tr} \left\{ P_- [\partial_s P_-, \delta_{\eta^0} P_-] \right\} + i \int_0^1 ds \operatorname{Tr} \left\{ P_+ [\partial_s P_+ \delta_{\eta^0} P_+] \right\} \\ &\quad + \int_0^1 ds \sum_{x \in \mathbb{Z}^4} \left\{ (\eta_\mu^{(1)0}(x) - \eta_{\mu[w]}^0(x)) \bar{k}_\mu(x) + \tilde{A}'_\mu(x) \delta_{\eta^0} \bar{k}_\mu(x) \right\}. \end{aligned} \quad (4.58)$$

One can see from this expression that $\mathfrak{K}_\eta^\diamond$ is defined in infinite volume for the variational parameter with a compact support $\eta_\mu^0(x)$. Then the following lemma holds true:

Lemma 4. $\mathfrak{K}_\eta^\diamond$ is in the form

$$\mathfrak{K}_\eta^\diamond = \mathfrak{L}_{\eta^0}^*, \quad (4.59)$$

where \mathfrak{L}_η^* is the linear functional defined in infinite volume for any variation parameter $\eta_\mu(x)$ with a compact support given by

$$\begin{aligned} \mathfrak{L}_\eta^* &= i \int_0^1 ds \left[\text{Tr} \{ P_- [\partial_s P_-, \delta_\eta P_-] \} + \text{Tr} \{ P_+ [\partial_s P_+, \delta_\eta P_+] \} \right]_{U_s = U^{(2)} \otimes e^{is\tilde{A}_\mu}} \\ &\quad + \int_0^1 ds \left[\sum_{x \in \mathbb{Z}^4} \left\{ \eta_\mu^{(1)}(x) \bar{k}_\mu(x) + \tilde{A}_\mu(x) \delta_\eta \bar{k}_\mu(x) \right\} \right]_{U_s = U^{(2)} \otimes e^{is\tilde{A}_\mu}} \\ &\equiv \sum_{x \in \mathbb{Z}^4} \{ \eta_\mu^a(x) j_\mu^{a*}(x) + \eta_\mu(x) j_\mu^*(x) \}. \end{aligned} \tag{4.60}$$

$\tilde{A}_\mu(x)$ here is the vector potential (in infinite volume) which represents the U(1) link field in the topological sector $\mathfrak{U}^{(1)}[0]$ (periodic in infinite volume), with the following properties,

$$\begin{aligned} U^{(1)}(x, \mu) &= e^{i\tilde{A}_\mu(x)}, & |\tilde{A}_\mu(x)| &\leq \pi(1 + 4\|x\|), \\ F_{\mu\nu}(x) &= \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) \end{aligned} \tag{4.61}$$

and any other field with these properties is equal to $\tilde{A}_\mu(x) + \partial_\mu \omega(x)$, where the gauge function $\omega(x)$ takes values that are integer multiples of 2π .

The proof of this lemma has been given for the U(1) case in our previous work [35] and it applies to the SU(2) \times U(1) case here simply by regarding the SU(2) link field as a background. So we omit it here.

The currents $j_\mu^{a*}(x)$ and $j_\mu^*(x)$ are quite similar in construction to $j_\mu^*(x)$ defined in [18] for the U(1) case. In particular, they are invariant under the gauge transformations $\tilde{A}_\mu(x) \rightarrow \tilde{A}_\mu(x) + \partial_\mu \omega(x)$ for arbitrary gauge functions $\omega(x)$ that are polynomially bounded at infinity. Then, the locality property of $j_\mu^{a*}(x)$ and $j_\mu^*(x)$ with respect to the U(1) link field can be established by the same argument as that given in [18]. The locality property with respect to the SU(2) link field follows from the locality property of the kernels of projection operators $P_\pm(x, y)$ and the current $\bar{k}_\mu(x)$.

5. Measure term in infinite volume

We note that \mathfrak{L}_η^* defined by eq. (4.60) provides the measure term of the lattice Glashow-Weinberg-Salam model in infinite volume. This non-perturbative result should be compared with the construction of the measure term in the weak coupling expansion [27]. One can see through the weak coupling expansion of eq. (4.60) that the result of [27] is automatically reproduced for the case of the SU(2) \times U(1) chiral gauge theory. Thus it provides a gauge-invariant lattice regularization of the Glashow-Weinberg-Salam model to all orders of perturbation theory.

6. Discussion

In this paper, we have given a gauge-invariant and non-perturbative construction of the Glashow-Weinberg-Salam model on the lattice, based on the lattice Dirac operator satisfying the Ginsparg-Wilson relation. We have shown that it is indeed possible to construct

the fermion measure of quarks and leptons which depends smoothly on the $SU(2) \times U(1)$ gauge fields and fulfills the fundamental requirements such as locality, gauge-invariance and lattice symmetries in all $SU(2)$ topological sectors with vanishing $U(1)$ magnetic flux. Then this construction would be usable for the studies of non-perturbative aspects of the Glashow-Weinberg-Salam model, such as the baryon number non-conservation. However, it is still desirable to extend our result in this paper to the topological sectors with non-vanishing $U(1)$ magnetic fluxes.

The measure term for the $SU(2) \times U(1)$ chiral gauge theory of the Glashow-Weinberg-Salam model may be constructed by solving the local cohomology problem formulated in 4+2 dimensions for generic non-abelian gauge theories [19, 24, 85]. The problem has been solved only in the infinite volume limit so far [28]. The measure term obtained in this paper provides an explicit solution to the 4+2 dimensional local cohomology problem in the finite volume for the topological sectors with vanishing $U(1)$ magnetic fluxes.

As for the formulation in the infinite volume, one may adopt the non-compact formulation for the $U(1)$ gauge theory, as discussed by Neuberger in [23]. Even for this case, the expression of the measure term given by eq. (4.60) holds true, if the vector potential there is identified as the dynamical field variables in the non-compact $U(1)$ formulation.

Towards a numerical application of the $SU(2) \times U(1)$ chiral lattice gauge theory of the Glashow-Weinberg-Salam model, the next step is the practical implementation of the formula of the chiral bases, eqs. (3.42)–(3.45): a computation of W and the implementation of the operator $Q_{t\pm}$. This question has been addressed partly for the $U(1)$ case in our previous works [34, 58, 59]. We will discuss this question in detail elsewhere.

Acknowledgments

Y.K. would like to thank Ting-Wai Chiu for his kind hospitality at 2005 Taipei Summer Institute on Strings, Particles and Fields and D. Adams, K. Fujikawa, H. Suzuki for discussions. Y.K. is also grateful to Y. Kuramashi for his nice organization of ILFTN workshop on "Perspectives in Lattice QCD" (Nara, 2005). Some of the proofs given in this paper were completed during this workshop. Y.K. is supported in part by Grant-in-Aid for Scientific Research No. 17540249.

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